THE COMPLETE FORM of N=2 SUPERGRAVITY and its PLACE in THE GENERAL FRAMEWORK of D=4 N–EXTENDED SUPERGRAVITIES

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Relying on the geometrical set up of Special Kähler Geometry and Quaternionic Geometry, which I discussed at length in my Lectures at the 1995 edition of this Spring School, I present here the recently obtained fully general form of N=2 supergravity with completely arbitrary couplings. This lagrangian has already been used in the literature to obtain various results: notably the partial breaking of supersymmetry and various extremal black–hole solutions. My emphasis, however, is only on providing the reader with a completely explicit and ready to use component expression of the supergravity action. All the details of the derivation are omitted but all the definitions of the items entering the lagrangian and the supersymmetry transformation rules are given.

1. Introduction

As a consequence of the recent exciting developments on the non–perturbative regimes of string theory, it is by now clear that all N–extended supergravities in the diverse dimensions from $D = 11$ to $D = 4$, constitute the low energy effective actions for a connected web of theories, describing both elementary and solitonic extended objects (the p–branes), that are related to each other by a complicated pattern of dualities, usually classified under the names of S, T and U [1]. These are generalizations of the electric–magnetic duality transformations of Maxwell theory and are just mere consequences of the remarkable geometric structure displayed by the scalar sector of supergravity theories. This structure has been known in the literature for already ten to fifteen years. In the last year edition of the Trieste Spring School I had the opportunity to give a series of lectures on the general form of these electric–magnetic duality rotations and on their bearing on the geometric structure of supergravity lagrangians [2]. As I emphasized there, the key instrument to understand the full interaction structure of supergravity theory is indeed the differential geometry of the scalar field sector and its symplectic embedding. Within the general framework, my main concern was the description of the peculiar geometric structures displayed by $N = 2$ supergravity, namely the *Special Kähler Geometry* pertaining to the vector multiplet sector and the *Quaternionic Geometry* pertaining to the hypermultiplet sector. Although the focus of those lectures was supergravity, no explicit mention was there given of the fermion fields, the purpose being the description of the bosonic sector geometry. In the months elapsed from last year lectures a fully general symplectic covariant formulation of $N = 2$ supergravity based on arbitrary special Kähler manifolds and arbitrary quaternionic manifolds with the gauging of an arbitrary group has become available [3]. This formulation includes all previous formulations obtained both by means of the conformal tensor calculus [4] and by means of the rheonomic approach [5], [6] but extends them to the most general situation. Notably within this formulation one can accommodate cases that were out of reach of previous formulations (for instance those where the special Kähler geometry admits no prepotential $F(X)$) and which are physically particularly relevant. Applications of this new formulation have already appeared in the literature. In particular in [7], by extending results obtained in

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[8], it has been shown that $N=2$ supersymmetry can spontaneously break to $N=1$ with a surviving unbroken compact gauge group.

In the present seminar it is my purpose to present the complete most general form of the N=2 supergravity lagrangian and of the supersymmetry transformation rules against which it is invariant.

2. How N=2 supergravity fits in the general scheme

Restricting my attention to the $D=4$ theories, I can recall from [2] that in their bosonic part all N–extended supergravity lagrangians admit the following general form:

$$\begin{aligned}
&\mathcal{L}^{SUGRA}_{Bose} = \\
&\sqrt{-g}\Big(R + g_{IJ}(\phi)\,\nabla_\mu\phi^I\,\nabla^\mu\phi^J \\
&+\mathrm{i}\,\big(\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{F}^{-\Sigma|\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\Sigma|\mu\nu}\big) \\
&-\mathcal{V}(\phi)\Big)
\end{aligned} \tag{1}$$

where ϕ^I denote the n_S scalar fields of the theory and

$$\mathcal{F}^{\pm\Lambda}_{\mu\nu} = \frac{1}{2}\left(F^\Lambda_{\mu\nu} \pm \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\Lambda|\mu\nu}\right) \tag{2}$$

denote the (anti)self–dual combination of the n_V field strengths:

$$\mathcal{F}^\Lambda_{\mu\nu} \equiv \frac{1}{2}\Big(\partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu + g f^\Lambda_{\Sigma\Delta}\,A^\Sigma_\mu\,A^\Delta_\nu\Big) \tag{3}$$

Indeed, whether pure or matter coupled, each supergravity theory is primarily characterized by these two numbers, *i.e.* the total number of scalars and the total number of vector fields. One has:

$$\begin{aligned}
n_S &= \#\ \text{scal. in scal. mult.} \\
&\quad + \#\ \text{scal. in vect. mult.} \\
&\quad + \#\ \text{scal. in grav. mult.} \\
n_V &= \#\ \text{scal. in vect. mult.} \\
&\quad + \#\ \text{scal. in grav. mult.}
\end{aligned} \tag{4}$$

and the available choices are summarized in table 1. To continue the illustration of eq. 1, by $\nabla_\mu\phi^I$ I denote the derivatives of the scalar fields, covariant with respect to the action of the gauge group G_{gauge}:

$$\nabla_\mu\phi^I = \partial\phi^I + g\,A^\Lambda_\mu\,k^I_\Lambda(\phi) \tag{5}$$

The Killing vector fields, appearing in eq.5

$$\vec{\mathbf{k}}_\Lambda \equiv k^I_\Lambda(\phi)\,\frac{\partial}{\partial\phi^I} \tag{6}$$

satisfy the Lie algebra $\mathbf{G}_{gauge}$ of the gauge group:

$$\left[\vec{\mathbf{k}}_\Lambda\,,\,\vec{\mathbf{k}}_\Sigma\right] = f^\Delta_{\Lambda\Sigma}\,\vec{\mathbf{k}}_\Delta \tag{7}$$

whose dimension is less or equal to the number of vector fields n_V. This algebra is a subalgebra of the isometry algebra of the scalar metric:

$$\mathbf{G}_{gauge} \subset \mathbf{G}_{iso} \tag{8}$$

By definition $\mathbf{G}_{iso}$ is generated by all those vector fields $\vec{\mathbf{t}} \in T\mathcal{M}_{scalar}$ such that:

$$\ell_{\vec{\mathbf{t}}}\,g_{IJ} = 0 \tag{9}$$

the operator ℓ denoting the Lie derivative. Hence eq.9 is in particular true when $\vec{\mathbf{t}} = \vec{\mathbf{k}}_\Lambda$. As extensively discussed in [2], the complex symmetric matrix $\mathcal{N}_{\Lambda\Sigma}(\phi)$, whose real and imaginary part play, respectively, the role of field dependent *theta*–angle and field dependent coupling constant, is determined by the symplectic embedding of the isometry group G_{iso}:

$$\iota_\delta\,:\,G_{iso} \longrightarrow Sp(2\,n_V,\mathrm{I\!R}) \tag{10}$$

so that for each element $\xi \in G_{iso}$ we have:

$$\begin{gathered}
\xi : \mathcal{M}_{scalar} \longrightarrow \mathcal{M}_{scalar} \\
\forall \vec{X},\vec{Y} \in T\mathcal{M}_{scalar} : \\
g\Big(\xi^\star\vec{X},\xi^\star\vec{Y}\Big) = g\Big(\vec{X},\vec{Y}\Big) \\
\mathcal{N}\left(\xi\left(\phi\right)\right) = \left(C_\xi + D_\xi\mathcal{N}\right)\left(A_\xi + B_\xi\mathcal{N}\right)^{-1}
\end{gathered} \tag{11}$$

where

$$\iota_\delta\left(\xi\right) = \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} \in Sp(2n_V,\mathrm{I\!R}) \tag{12}$$

is the symplectic image of ξ. When the scalar manifold is a homogeneous symmetric coset manifold, as it happens in all $N \geq 3$ supergravities

(see Table 1), the matrix $\mathcal{N}_{\Lambda\Sigma}$ is determined by a universal formula that was derived long ago by Gaillard and Zumino [9]. Taking into account the isomorphism:

$$\begin{aligned} Sp(2n_V, \mathbb{R}) &\sim Usp(n_V, n_V) \equiv \\ &Sp(2n_V, \mathbb{C}) \cap U(n_V, n_V) \end{aligned} \tag{13}$$

that is explicitly realized by the map:

$$\begin{aligned} \mu \,:\, Sp(2n_V, \mathbb{R}) \,\ni\, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \\ \begin{pmatrix} T & V^\star \\ V & T^\star \end{pmatrix} \in Usp(n_V, n_V) \end{aligned} \tag{14}$$

where:

$$\begin{aligned} T &= \frac{1}{2}\,(A - \mathrm{i}B) + \frac{1}{2}\,(C + \mathrm{i}D) \\ V &= \frac{1}{2}\,(A - \mathrm{i}B) - \frac{1}{2}\,(C + \mathrm{i}D) \end{aligned} \tag{15}$$

the symplectic embedding is such that, for $M_{scalar} = G_{iso}/H$ we have:

$$\begin{aligned} \mu \,\cdot\, \iota_\delta\,(G_{iso}) &\subset Usp(n_V, n_V) \\ \mu \,\cdot\, \iota_\delta\,(H) &\subset U(n_V) \subset Usp(n_V, n_V) \end{aligned} \tag{16}$$

Then, considering an arbitrary parametrization of the coset G_{iso}/H *i.e.*:

$$G_{iso}/H \,\ni\, \phi \,\longrightarrow\, L(\phi) \,\in\, G \tag{17}$$

by setting:

$$\begin{aligned} Usp(n_V, n_V) \,\ni\, \mathcal{O}(\phi) \,\equiv\, \mu \,\cdot\, \iota_\delta\,(L(\phi)) = \\ = \begin{pmatrix} U_0(\phi) & U_1^\star(\phi) \\ U_1(\phi) & U_0^\star(\phi) \end{pmatrix} \end{aligned} \tag{18}$$

we immediately obtain an immersion of G_{iso}/H into $Usp(n_V, n_V)/U(n_V)$ and the matrix $\mathcal{N}$ is obtained by defining, according to [9]:

$$\mathcal{N} \,\equiv\, \mathrm{i}\left[U_0^\dagger + U_1^\dagger\right]^{-1}\,\left[U_0^\dagger - U_1^\dagger\right] \tag{19}$$

The peculiarity of $N = 1$ and $N = 2$ supergravity with respect to the higher theories is that the scalar manifold is not necessarily a homogeneous symmetric coset manifold. Indeed the only request imposed by $N = 2$ supersymmetry on the scalar manifold M_{scalar} is that it should be the direct product:

$$M_{scalar} \,=\, \mathcal{SM}_n \,\otimes\, \mathcal{HM}_m \tag{20}$$

of a special Kähler manifold $\mathcal{SK}_n$ of complex dimension $\dim_{\mathbf{C}} \mathcal{SM}_n = n$ equal to the number of vector multiplets with a quaternionic manifold of quaternionic dimension $\dim_{\mathbf{Q}} \mathcal{HM}_m = m$ equal to the number of hypermultiplets. Correspondingly the scalar metric is of the form:

$$\begin{aligned} g_{IJ}(\phi)\, d\phi^I \otimes d\phi^J \quad &\equiv \quad g_{ij^\star}\, dz^i \otimes d\overline{z}^{j^\star} \\ &\qquad + h_{uv}\, dq^u \,\otimes\, dq^v \end{aligned} \tag{21}$$

where $g_{ij^\star}$ is the special Kähler metric on $\mathcal{SM}_n$ and h_{uv} is the quaternionic metric on $\mathcal{HM}_m$. Following the discussion of [2] there exists a symplectic holomorphic vector bundle with structural group $Sp(2n+2, \mathbb{R})$ and a section

$$\Omega = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \tag{22}$$

such that the Kähler potential can be written as:

$$\begin{aligned} \mathcal{K} \quad &= \quad -\log\left(\mathrm{i}\langle \Omega \,|\, \overline{\Omega}\rangle\right) \\ &= -\log\left[\mathrm{i}\left(\overline{X}^\Lambda F_\Lambda - \overline{F}_\Sigma X^\Sigma\right)\right] \end{aligned} \tag{23}$$

Defining furthermore

$$V \,=\, \left(\begin{array}{c} L^\Lambda \\ M_\Sigma \end{array}\right) \equiv\, e^{\mathcal{K}/2}\Omega \,=\, e^{\mathcal{K}/2}\left(\begin{array}{c} X^\Lambda \\ F_\Sigma \end{array}\right) \tag{24}$$

one has

$$\nabla_{i^\star} V \,=\, \left(\partial_{i^\star} - \frac{1}{2}\partial_{i^\star}\mathcal{K}\right)\, V \,=\, 0 \tag{25}$$

and setting:

$$U_i = \nabla_i V = \left(\partial_i + \frac{1}{2}\partial_i\mathcal{K}\right)\, V \equiv \left(\begin{array}{c} f_i^\Lambda \\ h_{\Sigma|i} \end{array}\right) \tag{26}$$

it follows that:

$$\nabla_i U_j = \mathrm{i}C_{ijk}\, g^{k\ell^\star}\, \overline{U}_{\ell^\star} \tag{27}$$

where ∇_i denotes the covariant derivative containing both the Levi–Civita connection on the bundle $\mathcal{TM}$ and the canonical connection θ on the line bundle $\mathcal{L}$ whose first Chern class equal the Kähler class. In eq. 27 the symbol C_{ijk} denotes a covariantly holomorphic ($\nabla_{\ell^\star} C_{ijk} = 0$) section of the bundle $\mathcal{TM}^3 \otimes \mathcal{L}^2$ that is totally symmetric in its indices. It enters in the construction of the lagrangian and of the transformation

rules together with the upper part L^Λ of the non–holomorphic symplectic section V and the upper part of its derivative f_i^Λ. The matrix $\mathcal{N}$ is defined by the identity:

$$\overline{M}_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma}\overline{L}^\Sigma \quad ; \quad h_{\Sigma|i} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma \tag{28}$$

In the $N=2$ case, the Killing vector $\vec{\mathbf{k}}_\Lambda$ is composed of two parts.

$$\vec{\mathbf{k}}_\Lambda = \left[k^i_\Lambda \frac{\partial}{\partial z^i} + k^{i^\star}_\Lambda \frac{\partial}{\partial z^{i^\star}}\right] + k^u_\Lambda \frac{\partial}{\partial q^u} \tag{29}$$

corresponding to the infinitesimal action of the gauge group on the special Kähler manifold and on the quaternionic manifold, respectively. Naming K^x the triplet of HyperKähler 2–forms that, by definition of quaternionic manifold, do necessarily exist on $\mathcal{HM}$, we introduce the notion of triholomorphic momentum map associating to each killing vector $k^u_\Lambda \frac{\partial}{\partial q^u}$ a triplet of 0–form prepotentials $\mathcal{P}^z_\Lambda$ defined through the following equation:

$$\mathbf{i}_\Lambda K^x = -\nabla \mathcal{P}^x_\Lambda \equiv -(d\mathcal{P}^x_\Lambda + \epsilon^{xyz}\omega^y \mathcal{P}^z_\Lambda) \tag{30}$$

The components of the HyperKähler 2–forms satisfy the quaternionic algebra:

$$h^{st} K^x_{us} K^y_{tw} = -\delta^{xy} h_{uw} + \epsilon^{xyz} K^z_{uw} \tag{31}$$

In addition one needs also the quaternionic vielbein $\mathcal{U}^{A\alpha}$ that is a 1–form with a pair of tangent indices, the first (A) taking two values and transforming in the doublet representation of $SU(2)$, the second (α) taking $2m$ values and transforming in the fundamental representation of $Sp(2m)$. The relation between the quaternionic vielbein, the quaternionic metric and the triplet of HyperKähler 2–forms is given by:

$$\begin{aligned}
(\mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v + \mathcal{U}^{A\alpha}_v \mathcal{U}^{B\beta}_u)\mathbb{C}_{\alpha\beta} &= h_{uv}\epsilon^{AB} \\
(\mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v + \mathcal{U}^{A\alpha}_v \mathcal{U}^{B\beta}_u)\epsilon_{AB} &= h_{uv}\frac{1}{m}\mathbb{C}^{\alpha\beta} \\
\mathrm{i}\,\mathbb{C}_{\alpha\beta}(\sigma_x)_A{}^C \epsilon_{CB} \mathcal{U}^{\alpha A} \wedge \mathcal{U}^{\beta B} &= K^x
\end{aligned} \tag{32}$$

where

$$\mathcal{U}_{A\alpha} \equiv (\mathcal{U}^{A\alpha})^* = \epsilon_{AB}\mathbb{C}_{\alpha\beta}\mathcal{U}^{B\beta} \tag{33}$$

and where ϵ_{AB} denotes the Levi–Civita tensor in 2–dimensions and $\mathbb{C}_{\alpha\beta}$ is the symplectic invariant constant antisymmetric matrix.

To complete the list of geometrical data entering the supergravity lagrangian one still needs the $SU(2)\otimes Sp(2m)$ Lie algebra valued spin connection of $\mathcal{HM}$ that is implicitly defined by the vanishing torsion equation:

$$\begin{aligned}
\nabla \mathcal{U}^{A\alpha} &\equiv d\mathcal{U}^{A\alpha} + \frac{i}{2}\omega^x (\epsilon\sigma_x\epsilon^{-1})^A{}_B \wedge \mathcal{U}^{B\alpha} \\
&+ \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma}\mathbb{C}_{\beta\gamma} = 0
\end{aligned} \tag{34}$$

3. The fermion fields

In order to present the full supergravity lagrangian we need to introduce the fermion fields. They are

- the gravitino:

  $$\psi_A \equiv \psi_{A\mu}\, dx^\mu \tag{35}$$

  which is an $SU(2)$ doublet of chiral ($\gamma_5\,\psi_A = \psi_A$) spinor valued 1–forms (the complex conjugate doublet corresponding to the opposite chiral projection : $\gamma_5\,\psi_A = \psi_A$)

- the gauginos:

  $$\gamma_5\,\lambda^{iA} = \lambda^{iA} \quad ; \quad \gamma_5\,\lambda^{i^\star}_A = -\lambda^{i^\star}_A \tag{36}$$

  which, besides being 4–dimensional spinors, transform as world tensors on the special Kähler manifold and as sections of the Hodge line–bundle

- the hyperinos

  $$\gamma_5\,\zeta^\alpha = -\zeta^\alpha \quad ; \quad \gamma_5\,\zeta_\alpha = \zeta_\alpha \tag{37}$$

  which, besides being 4–dimensional spinors, transform in the fundamental representation of the $Sp(2m)$ holonomy group of $\mathcal{HM}$

Then the covariant derivatives of the fermion fields are:

$$\begin{aligned}
\rho_A &\equiv d\psi_A - \frac{1}{4}\gamma_{ab}\,\omega^{ab}\wedge\psi_A \\
&\quad + \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\wedge\psi_A + \widehat{\omega}_A{}^B\wedge\psi_B \\
\rho^A &\equiv d\psi^A - \frac{1}{4}\gamma_{ab}\,\omega^{ab}\wedge\psi^A
\end{aligned}$$

$$
\begin{aligned}
& -\frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\wedge\psi^A+\widehat{\omega}^A{}_B\wedge\psi^B \\
\nabla\lambda^{iA} &\equiv d\lambda^{iA}-\frac{1}{4}\gamma_{ab}\,\omega^{ab}\lambda^{iA} \\
& -\frac{\mathrm{i}}{2}\widehat{Q}\lambda^{iA}+\widehat{\Gamma}^i{}_j\lambda^{jA}+\widehat{\omega}^A_B\wedge\lambda^{iB} \\
\nabla\lambda^{i^\star}_A &\equiv d\lambda^{\bar{\imath}}_A-\frac{1}{4}\gamma_{ab}\,\omega^{ab}\lambda^{i^\star}_A \\
& +\frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\lambda^{i^\star}_A+\widehat{\Gamma}^{i^\star}{}_{j^\star}\lambda^{j^\star}_A+\widehat{\omega}_A{}^B\wedge\lambda^{i^\star}_B \\
\nabla\zeta_\alpha &\equiv d\zeta_\alpha\,-\,\frac{1}{4}\omega^{ab}\,\gamma_{ab}\,\zeta_\alpha \\
& -\frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\,\zeta_\alpha\,+\widehat{\Delta}_\alpha{}^\beta\zeta_\beta \\
\nabla\zeta^\alpha &\equiv d\zeta^\alpha\,-\,\frac{1}{4}\omega^{ab}\,\gamma_{ab}\,\zeta^\alpha \\
& +\frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\,\zeta^\alpha\,+\widehat{\Delta}^\alpha{}_\beta\zeta^\beta
\end{aligned}
\tag{38}
$$

In the above definitions the connections acting on the fermion fields are the gauged ones, namely:

$$
\begin{array}{lcl}
\Gamma^i{}_j & \rightarrow & \widehat{\Gamma}^i{}_j=\Gamma^i{}_j+g\,A^\Lambda\,\partial_j k^i_\Lambda \\
\mathcal{Q} & \rightarrow & \widehat{\mathcal{Q}}=\mathcal{Q}-ig\,A^\Lambda\,k^i_\Lambda\,\partial_i\mathcal{K} \\
\omega^x & \rightarrow & \widehat{\omega}^x=\omega^x+g\,A^\Lambda\,\mathcal{P}^x_\Lambda \\
\Delta^{\alpha\beta} & \rightarrow & \widehat{\Delta}^{\alpha\beta}=\Delta^{\alpha\beta} \\
 & & +g\,A^\Lambda\,\partial_u k^v_\Lambda\,\mathcal{U}^{u|\alpha A}\,\mathcal{U}^\beta_{v|A}
\end{array}
\tag{39}
$$

4. The Lagrangian and the supersymmetry transformation rules

In terms of the geometrical data so far introduced and of the covariant derivatives defined above the complete N=2 supergravity Lagrangian can now be presented. It is displayed in table 2. To make the formula completely explicit it suffices to recall that by $(...)^-$ I have denoted the self dual part of any fermion bilinear combination involving the $\gamma_{\mu\nu}$ matrix. Furthermore the explicit form of the fermion mass–matrices appearing in the lagrangian is given below:

N=2 Supergravity mass matrices

$$
\begin{aligned}
S_{AB} &= \frac{\mathrm{i}}{2}(\sigma_x)_A{}^C\epsilon_{BC}\mathcal{P}^x_\Lambda L^\Lambda \\
W^{iAB} &= \epsilon^{AB}\,k^i_\Lambda\,\overline{L}^\Lambda\,+ \\
& \quad \mathrm{i}(\sigma_x)_C{}^B\epsilon^{CA}\mathcal{P}^x_\Lambda g^{ij^\star}\,\overline{f}^\Lambda_{j^\star} \\
N^A_\alpha &= 2\,\mathcal{U}^A_{\alpha|u}\,k^u_\Lambda\,\overline{L}^\Lambda \\
\mathcal{M}^{\alpha|\beta} &= -\,g\,\mathcal{U}^{\alpha A}_u\,\mathcal{U}^{\beta B}_v\,\varepsilon_{AB}\,\nabla^{[u}k^{v]}_\Lambda\,L^\Lambda \\
\mathcal{M}^{\alpha|}{}_{|iB} &= -4\,g\,\mathcal{U}^\alpha_{B|u}\,k^u_\Lambda\,f^\Lambda_i \\
\mathcal{M}_{iA|\ell B} &= \frac{1}{3}\,g\,\Big(\varepsilon_{AB}\,g_{ij^\star}\,k^{j^\star}_\Lambda\,f^\Lambda_\ell \\
& \quad +\mathrm{i}(\sigma_x)_A{}^C\epsilon_{BC}\,\mathcal{P}^x_\Lambda\,\nabla_\ell f^\Lambda_i\Big)
\end{aligned}
\tag{40}
$$

The coupling constant in front of the mass–matrices and of the potential is just a symbolic notation to remind the reader that these terms are entirely due to the gauging and vanish in the ungauged theory. In general there is not a single coupling constant rather there are as many independent coupling constants as the number of factors in the gauge group. This fact has been exploited in [7,8] to obtain the partial supersymmetry breaking $N=2\longrightarrow N=1$ by gauging a non compact abelian group with appropriate ratio of the coupling constants associated with each generator.

5. The supersymmetry transformation rules

Of prominent interest in many applications is, besides that of the action, the form of the supersymmetry transformation rules. For instance this information is essential in order to obtain the differential equations describing BPS saturated states [10],[11],[12], [13],[15], namely those field configurations that preserve 1/2 of the original supersymmetries. Another respect where the supersymmetry transformation rules are of vital importance is in the topological twist to topological field theories [16], [17], [18], [19].

The explicit form of the SUSY rules is listed below

Supergravity transformation rules of the Fermi fields

$$\delta\,\Psi_{A|\mu} =$$
$$\mathcal{D}_\mu\,\epsilon_A\,-\tfrac{1}{4}\left(\partial_i\,\mathcal{K}\overline{\lambda}^{iB}\,\epsilon_B\,-\,\partial_{i^\star}\,\mathcal{K}\overline{\lambda}^{i^\star}_B\epsilon^B\right)\Psi_{A|\mu}$$
$$-\omega_{A|v}^{\ \ B}\,\mathcal{U}^v_{C\alpha}\,\left(\epsilon^{CD}\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_\beta\,\epsilon_D\,+\,\overline{\zeta}^\alpha\,\epsilon^C\right)\Psi_{B|\mu}$$
$$+\,\left(A_A^{\ |\nu B}\eta_{\mu\nu}+A'^{\ |\nu B}_A\gamma_{\mu\nu}\right)\epsilon_B$$
$$+\left[\mathrm{i}\,g\,S_{AB}\eta_{\mu\nu}+\epsilon_{AB}(T^-_{\mu\nu}\,+U^+_{\mu\nu})\right]\gamma^\nu\epsilon^B \tag{41}$$

$$\delta\,\lambda^{iA} =$$
$$\tfrac{1}{4}\left(\partial_j\,\mathcal{K}\overline{\lambda}^{jB}\,\epsilon_B\,-\,\partial_{j^\star}\,\mathcal{K}\overline{\lambda}^{j^\star}_B\epsilon^B\right)\lambda^{iA}$$
$$-\omega^A_{\ \ B|v}\,\mathcal{U}^v_{C\alpha}\,\left(\epsilon^{CD}\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_\beta\,\epsilon_D\,+\,\overline{\zeta}^\alpha\,\epsilon^C\right)\lambda^{iB}$$
$$-\,\Gamma^i_{\ jk}\overline{\lambda}^{kB}\,\epsilon_B\,\lambda^{jA}+$$
$$\mathrm{i}\,\left(\nabla_\mu\,z^i-\overline{\lambda}^{iA}\,\psi_{A|\mu}\right)\gamma^\mu\epsilon^A$$
$$+G^{-i}_{\mu\nu}\gamma^{\mu\nu}\epsilon_B\epsilon^{AB}\,+\,D^{iAB}\epsilon_B \tag{42}$$

$$\delta\,\zeta_\alpha =$$
$$-\Delta_{\alpha|v}^{\ \ \beta}\,\mathcal{U}^v_{\gamma A}\,\left(\epsilon^{AB}\,\mathbb{C}^{\gamma\delta}\,\overline{\zeta}_\delta\,\epsilon_B\,+\,\overline{\zeta}^\gamma\,\epsilon^A\right)\zeta_\beta$$
$$+\tfrac{1}{4}\left(\partial_i\,\mathcal{K}\overline{\lambda}^{iB}\,\epsilon_B\,-\,\partial_{i^\star}\,\mathcal{K}\overline{\lambda}^{i^\star}_B\epsilon^B\right)\zeta_\alpha$$
$$+\,\mathrm{i}\,\Big(\mathcal{U}^{B\beta}_u\,\nabla_\mu\,q^u$$
$$-\epsilon^{BC}\,\mathbb{C}^{\beta\gamma}\,\overline{\zeta}_\gamma\,\psi_C\,-\,\overline{\zeta}^\beta\,\psi^B\Big)\,\gamma^\mu\epsilon^A\epsilon_{AB}\,\mathbb{C}_{\alpha\beta}$$
$$+\,g\,N^A_\alpha\,\epsilon_A \tag{43}$$

Supergravity transformation rules
of the Bose fields

$$\begin{aligned}
\delta\,V^a_\mu &= -\mathrm{i}\,\overline{\Psi}_{A|\mu}\,\gamma^a\,\epsilon^A-\mathrm{i}\,\overline{\Psi}^A_\mu\,\gamma^a\,\epsilon_A\\
\delta\,A^\Lambda_\mu &= 2\overline{L}^\Lambda\overline{\psi}_{A|\mu}\epsilon_B\epsilon^{AB}\,+\,2L^\Lambda\overline{\psi}^A_\mu\epsilon^B\epsilon_{AB}\\
&\quad +\Big(\mathrm{i}\,f^\Lambda_i\,\overline{\lambda}^{iA}\gamma_\mu\epsilon^B\,\epsilon_{AB}\\
&\quad +\mathrm{i}\,\overline{f}^\Lambda_{i^\star}\,\overline{\lambda}^{i^\star}_A\gamma_\mu\epsilon_B\,\epsilon^{AB}\Big)\\
\delta\,z^i &= \overline{\lambda}^{iA}\epsilon_A\\
\delta\,z^{i^\star} &= \overline{\lambda}^{i^\star}_A\epsilon^A\\
\delta\,q^u &= \mathcal{U}^u_{\alpha A}\left(\overline{\zeta}^\alpha\epsilon^A+\mathbb{C}^{\alpha\beta}\epsilon^{AB}\overline{\zeta}_\beta\epsilon_B\right)
\end{aligned} \tag{44}$$

In the above rules there appear certain field combinations which have been given special names. In an off–shell formulation of supergravity these would be auxiliary fields. Their explicit on–shell expression in terms of the fundamental physical fields is given below:

Supergravity values
of the auxiliary fields

$$\begin{aligned}
A_A^{\ |\mu B} &= -\frac{\mathrm{i}}{4}\,g_{k^\star\ell}\left(\overline{\lambda}^{k^\star}_A\,\gamma^\mu\lambda^{\ell B}\,-\right.\\
&\quad \left.\delta^B_A\,\overline{\lambda}^{k^\star}_C\,\gamma^\mu\lambda^{\ell C}\right)\\
A'^{\ |\mu B}_A &= \frac{\mathrm{i}}{4}\,g_{k^\star\ell}\left(\overline{\lambda}^{k^\star}_A\,\gamma^\mu\lambda^{\ell B}\,-\right.\\
&\quad \left.\frac{1}{2}\,\delta^B_A\,\overline{\lambda}^{k^\star}_C\,\gamma^\mu\lambda^{C\ell}\right)\,-\,\frac{\mathrm{i}}{4}\,\delta^B_A\,\overline{\zeta}_\alpha\gamma^\mu\zeta^\alpha
\end{aligned} \tag{45}$$

$$\begin{aligned}
T^-_{\mu\nu} &= \left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Lambda\Sigma}L^\Sigma\Big(\widetilde{F}^{\Lambda-}_{\mu\nu}\\
&\quad +\frac{1}{8}\nabla_i\,f^\Lambda_j\,\overline{\lambda}^{iA}\gamma_{\mu\nu}\,\lambda^{jB}\,\epsilon_{AB}\\
&\quad -\frac{1}{4}\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_\alpha\gamma_{\mu\nu}\,\zeta_\beta\,L^\Lambda\Big)\\
T^+_{\mu\nu} &= \left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Lambda\Sigma}\overline{L}^\Sigma\Big(\widetilde{F}^{\Lambda+}_{\mu\nu}\\
&\quad +\frac{1}{8}\nabla_{i^\star}\,\overline{f}^\Lambda_{j^\star}\,\overline{\lambda}^{i^\star}_A\gamma_{\mu\nu}\,\lambda^{j^\star}_B\,\epsilon^{AB}\\
&\quad -\frac{1}{4}\,\mathbb{C}_{\alpha\beta}\,\overline{\zeta}^\alpha\gamma_{\mu\nu}\,\zeta^\beta\,\overline{L}^\Lambda\Big)\\
U^-_{\mu\nu} &= -\frac{\mathrm{i}}{4}\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_\alpha\gamma_{\mu\nu}\,\zeta_\beta\\
U^+_{\mu\nu} &= -\frac{\mathrm{i}}{4}\,\mathbb{C}_{\alpha\beta}\,\overline{\zeta}^\alpha\gamma_{\mu\nu}\,\zeta^\beta
\end{aligned} \tag{46}$$

$$G^{i-}_{\mu\nu} = \frac{\mathrm{i}}{2}\,g^{ij^\star}\overline{f}^\Gamma_{j^\star}\,\left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Gamma\Lambda}\Big(\widetilde{F}^{\Lambda-}_{\mu\nu}$$

$$
\begin{aligned}
& +\frac{1}{8}\nabla_k f^\Lambda_\ell \overline{\lambda}^{kA}\gamma_{\mu\nu}\,\lambda^{\ell B}\epsilon_{AB} \\
& -\frac{1}{4}\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_\alpha\gamma_{\mu\nu}\,\zeta_\beta\,L^\Lambda\Big) \\
G^{i^\star +}_{\mu\nu} \quad = \quad & \frac{\mathrm{i}}{2}\,g^{i^\star j}f^\Gamma_j\left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Gamma\Lambda}\Big(\widetilde{F}^{\Lambda +}_{\mu\nu} \\
& +\frac{1}{8}\nabla_{k^\star}\overline{f}^\Lambda_{\ell^\star}\,\overline{\lambda}^{k^\star}_A\,\gamma_{\mu\nu}\,\lambda^{\ell^\star}_B\,\epsilon^{AB} \\
& -\frac{1}{4}\mathbb{C}_{\alpha\beta}\,\overline{\zeta}^\alpha\gamma_{\mu\nu}\,\zeta^\beta\,\overline{L}^\Lambda\Big) \\
D^{iAB} \quad = \quad & \frac{\mathrm{i}}{2}g^{ij^\star}\,C_{j^\star k^\star \ell^\star}\,\overline{\lambda}^{k^\star}_C\,\lambda^{\ell^\star}_D\,\epsilon^{AC}\epsilon^{BD} \\
& +W^{iAB}
\end{aligned}
\tag{47}
$$

In the above equations we have denoted by $\widetilde{F}$ the supercovariant field strength defined by:

$$
\begin{aligned}
\widetilde{F}^\Lambda_{\mu\nu} \quad = \quad & \mathcal{F}^\Lambda_{\mu\nu} + L^\Lambda\overline{\psi}^A_\mu\psi^B_\nu\,\epsilon_{AB} \\
& +\overline{L}^\Lambda\overline{\psi}_{A\mu}\psi_{B\nu}\epsilon^{AB} \\
& -\mathrm{i}\,f^\Lambda_i\,\overline{\lambda}^{iA}\gamma_{[\nu}\psi^B_{\mu]}\,\epsilon_{AB} \\
& -\mathrm{i}\,\overline{f}^\Lambda_{i^\star}\,\overline{\lambda}^{i^\star}_A\gamma_{[\nu}\psi_{B\mu]}\,\epsilon^{AB}
\end{aligned}
\tag{48}
$$

6. Final comments

Let us make some observation about the structure of the Lagrangian and of the transformation laws.

i) We note that all the terms of the Lagrangian are given in terms of purely geometric objects pertaining to either the special or the quaternionic geometry. Furthermore the Lagrangian does not rely on the existence of a prepotential function $F = F(X)$ and it is valid for any choice of the quaternionic manifold.

ii) The Lagrangian is not invariant under symplectic duality transformations. However, in absence of gauging ($g = 0$), if we restrict the Lagrangian to configurations where the vectors are on shell, it becomes symplectic invariant

iii) We note that the field strengths $\mathcal{F}^{\Lambda -}_{\mu\nu}$ originally introduced in the Lagrangian are the free gauge field strengths.The interacting field strengths which are supersymmetry eigenstates are defined as the objects appearing in the transformation laws of the gravitinos and gauginos fields,respectively,namely the bosonic part of $T^-_{\mu\nu}$ and $G^{-i}_{\mu\nu}$. In static configurations the integral of these objects on a 2–sphere at infinity define the central charge and the matter charges.

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Table 1
Scalar Manifolds of Extended Supergravities

N	# scal. in scal.m.	# scal. in vec. m.	# scal. in grav. m.	# vect. in vec. m.	# vect. in grav. m.	Γ_{cont}	$\mathcal{M}_{scalar}$
1	2 m			n		$\mathcal{I}$ $\subset Sp(2n, \mathbb{R})$	Kähler
2	4 m	2 n		n	1	$\mathcal{I}$ $\subset Sp(2n+2, \mathbb{R})$	Quaternionic $\otimes$ Special Kähler
3		6 n		n	3	$SU(3,n)$ $\subset Sp(2n+6, \mathbb{R})$	$\frac{SU(3,n)}{S(U(3)\times U(n))}$
4		6 n	2	n	6	$SU(1,1)\otimes SO(6,n)$ $\subset Sp(2n+12, \mathbb{R})$	$\frac{SU(1,1)}{U(1)}\otimes \frac{SO(6,n)}{SO(6)\times SO(n)}$
5			10		10	$SU(1,5)$ $\subset Sp(20, \mathbb{R})$	$\frac{SU(1,5)}{S(U(1)\times U(5))}$
6			30		16	$SO^\star(12)$ $\subset Sp(32, \mathbb{R})$	$\frac{SO^\star(12)}{U(1)\times SU(6)}$
$7,8$			70		56	$E_{7(-7)}$ $\subset Sp(128, \mathbb{R})$	$\frac{E_{7(-7)}}{SU(8)}$

Table 2:

The N=2 Supergravity action

$$S = \int \sqrt{-g}\, d^4 x \,(\mathcal{L}_{grav} + \mathcal{L}_{kin} + \mathcal{L}_{Pauli} + \mathcal{L}_{gauging} + \mathcal{L}_{4ferm})$$

$$\begin{aligned}
\mathcal{L}_{grav} &= -\frac{1}{2} R + \left(\overline{\Psi}^A_\mu \gamma_\sigma \rho_{A|\nu\lambda} - \overline{\Psi}_{A\mu} \gamma_\sigma \rho^A_{\nu\lambda}\right) \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}} \\
\mathcal{L}_{kin} &= g_{ij^\star} \nabla^\mu z^i \nabla_\mu \overline{z}^{j^\star} + h_{uv} \nabla_\mu q^u \nabla^\mu q^v \\
&\quad - \frac{\mathrm{i}}{2} g_{ij^\star} \left(\overline{\lambda}^{iA} \gamma^\mu \nabla_\mu \lambda^{j^\star}_A + \overline{\lambda}^{j^\star}_A \gamma^\mu \nabla_\mu \lambda^{iA}\right) - \mathrm{i} \left(\overline{\zeta}^\alpha \gamma^\mu \nabla_\mu \zeta_\alpha + \overline{\zeta}_\alpha \gamma^\mu \nabla_\mu \zeta^\alpha\right) \\
&\quad + \mathrm{i} \left(\overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{F}^{-\Lambda}_{\mu\nu} \mathcal{F}^{-\Sigma|\mu\nu} - \mathcal{N}_{\Lambda\Sigma} \mathcal{F}^{+\Lambda}_{\mu\nu} \mathcal{F}^{+\Sigma|\mu\nu}\right) \\
\mathcal{L}_{Pauli} &= \mathcal{L}^{inv}_{Pauli} + \mathcal{L}^{non\,inv}_{Pauli} \\
\mathcal{L}^{inv}_{Pauli} &= - g_{ij^\star} \left(\nabla_\mu \overline{z}^{j^\star} \overline{\Psi}^\mu_A \lambda^{iA} + h.c.\right) - 2 \left(\mathcal{U}^{A\alpha}_u \nabla_\mu q^u \overline{\Psi}^\mu_A \zeta_\alpha + h.c.\right) \\
&\quad + g_{ij^\star} \left(\nabla_\mu \overline{z}^{j^\star} \overline{\lambda}^{iA} \gamma^{\mu\nu} \Psi_{A|\nu} + h.c.\right) + 2 \left(\mathcal{U}^{\alpha A}_\mu \overline{\zeta}_\alpha \gamma^{\mu\nu} \Psi_{A|\nu} + h.c.\right) \\
\mathcal{L}^{non\,inv}_{Pauli} &= \mathcal{L}^{-\,non\,inv}_{Pauli} + \mathcal{L}^{+\,non\,inv}_{Pauli} \qquad \left(\mathcal{L}^{+\,non\,inv}_{Pauli} = \left(\mathcal{L}^{-\,non\,inv}_{Pauli}\right)^\star\right) \\
\mathcal{L}^{-\,non\,inv}_{Pauli} &= \mathcal{F}^{-\Lambda}_{\mu\nu} \left(\mathcal{N} - \overline{\mathcal{N}}\right)_{\Lambda\Sigma} [- 2\,\mathrm{i}\, L^\Sigma \overline{\Psi}^{A|\mu} \Psi^{B|\nu} \epsilon_{AB} - 2\, \overline{f}^\Sigma_{i^\star} \overline{\lambda}^{i^\star}_A \gamma^\nu \Psi^\mu_B \epsilon^{AB} \\
&\quad - \frac{\mathrm{i}}{4} \nabla_i f^\Sigma_j \overline{\lambda}^{iA} \gamma^{\mu\nu} \lambda^{jB} \epsilon_{AB} + \frac{\mathrm{i}}{2} L^\Sigma \overline{\zeta}_\alpha \gamma^{\mu\nu} \zeta_\beta C^{\alpha\beta}] \\
\mathcal{L}_{gauging} &= \mathcal{L}_{massmatrix} + \mathcal{L}_{potential} \\
\mathcal{L}_{massmatrix} &= \Bigg[2\, g\, S_{AB} \overline{\Psi}^A_\mu \gamma^{\mu\nu} \Psi^B_\nu + \mathrm{i}\, g\, g_{ij^\star} W^{iAB} \overline{\lambda}^{j^\star}_A \gamma_\mu \Psi^\mu_B + 2\,\mathrm{i}\, g\, N^A_\alpha \overline{\zeta}^\alpha \gamma_\mu \Psi^\mu_A \\
&\quad + \mathcal{M}^{\alpha|\beta} \overline{\zeta}_\alpha \zeta_\beta + \mathcal{M}^{\alpha|}{}_{|iB} \overline{\zeta}_\alpha \lambda^{iB} + \mathcal{M}_{iA|\ell B} \overline{\lambda}^{iA} \lambda^{\ell B} \Bigg] + \mathrm{h.c.} \\
\mathcal{L}_{potential} &= -\mathrm{V}(z, \overline{z}, q) = \\
&\quad - g^2 \Big[\left(g_{ij^\star} k^i_\Lambda k^{j^\star}_\Sigma + 4\, h_{uv} k^u_\Lambda k^v_\Sigma\right) \overline{L}^\Lambda L^\Sigma \\
&\quad + g^{ij^\star} f^\Lambda_i f^\Sigma_{j^\star} \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma - 3\, \overline{L}^\Lambda L^\Sigma \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma \Big] \\
\mathcal{L}_{4ferm} &= \mathcal{L}^{inv}_{4ferm} + \mathcal{L}^{non\,inv}_{4ferm} \\
\mathcal{L}^{inv}_{4ferm} &= \frac{\mathrm{i}}{2} \left(g_{ij^\star} \overline{\lambda}^{iA} \gamma_\sigma \lambda^{j^\star}_B - 2\, \delta^A_B \overline{\zeta}^\alpha \gamma_\sigma \zeta_\alpha\right) \overline{\Psi}_{A|\mu} \gamma_\lambda \Psi^B_\nu \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{6}\left(C_{ijk}\overline{\lambda}^{iA}\gamma^\mu\Psi^B_\mu\,\overline{\lambda}^{jC}\lambda^{kD}\,\epsilon_{AC}\epsilon_{BD}+h.c.\right)\\
& -2\overline{\Psi}^A_\mu\Psi^B_\nu\,\overline{\Psi}^\mu_A\Psi^\nu_B+2g_{ij^\star}\,\overline{\lambda}^{iA}\gamma_\mu\Psi^B_\nu\,\overline{\lambda}^{i^\star}_A\gamma^\mu\Psi^\nu_B\\
& +\frac{1}{4}\left(R_{ij^\star lk^\star}+g_{ik^\star}\,g_{lj^\star}-\frac{3}{2}\,g_{ij^\star}\,g_{lk^\star}\right)\overline{\lambda}^{iA}\lambda^{lB}\overline{\lambda}^{j^\star}_A\lambda^{k^\star}_B\\
& +\frac{1}{4}\,g_{ij^\star}\,\overline{\zeta}^\alpha\gamma_\mu\zeta_\alpha\,\overline{\lambda}^{iA}\gamma^\mu\lambda^{j^\star}_A+\frac{1}{2}\,\mathcal{R}^\alpha_{\beta ts}\,\mathcal{U}^t_{A\gamma}\,\mathcal{U}^s_{B\delta}\epsilon^{AB}\,C^{\delta\eta}\overline{\zeta}_\alpha\,\zeta_\eta\,\overline{\zeta}^\beta\,\zeta^\gamma\\
& -\left[\frac{\mathrm{i}}{12}\nabla_m\,C_{jkl}\overline{\lambda}^{jA}\lambda^{mB}\overline{\lambda}^{kC}\lambda^{lD}\epsilon_{AC}\epsilon_{BD}+h.c.\right]\\
& +g_{ij^\star}\,\overline{\Psi}^A_\mu\lambda^{j^\star}_A\,\overline{\Psi}^\mu_B\lambda^{iB}+2\overline{\Psi}^A_\mu\zeta^\alpha\overline{\Psi}^\mu_A\zeta_\alpha\\
& +\left(\epsilon_{AB}\,\mathbb{C}_{\alpha\beta}\,\overline{\Psi}^A_\mu\zeta^\alpha\,\overline{\Psi}^{B|\mu}\zeta^\beta+h.c.\right)
\end{aligned}
$$

$$
\mathcal{L}^{non\,inv}_{4ferm} = \mathcal{L}^{-\,non\,inv}_{4ferm}+\mathcal{L}^{+\,non\,inv}_{4ferm} \qquad \left(\mathcal{L}^{+\,non\,inv}_{4ferm}=\left(\mathcal{L}^{-\,non\,inv}_{4ferm}\right)^\star\right)
$$

$$
\begin{aligned}
\mathcal{L}^{-\,non\,inv}_{4ferm} = {} & \left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Lambda\Sigma}\Big[-\mathrm{i}\,L^\Lambda\,L^\Sigma\left(\overline{\Psi}^A_\mu\Psi^B_\nu\right)^-\left(\overline{\Psi}^C_\mu\Psi^D_\nu\right)^-\epsilon_{AB}\,\epsilon_{CD}\\
& -4\,L^\Lambda\overline{f}^\Sigma_{i^\star}\left(\overline{\Psi}^A_\mu\Psi^B_\nu\right)^-\left(\overline{\lambda}^{i^\star}_A\gamma^\nu\Psi^\mu_B\right)^-\\
& +\mathrm{i}\,\overline{f}^\Lambda_{i^\star}\overline{f}^\Sigma_{j^\star}\left(\overline{\lambda}^{i^\star}_A\gamma^\nu\Psi^\mu_B\right)^-\left(\overline{\lambda}^{j^\star}_C\gamma_\nu\Psi_{D|\mu}\right)^-\epsilon^{AB}\,\epsilon^{CD}\\
& +\frac{1}{4}L^\Lambda\overline{f}^\Sigma_{\ell^\star}\,g^{k\overline{\ell}}\,C_{ijk}\left(\overline{\Psi}^A_\mu\Psi^B_\nu\right)^-\overline{\lambda}^{iC}\gamma^{\mu\nu}\lambda^{jD}\,\epsilon_{AB}\,\epsilon_{CD}\\
& -\frac{\mathrm{i}}{2}\overline{f}^\Lambda_{\overline{m}}\overline{f}^\Sigma_{\ell^\star}\,g^{k\overline{\ell}}\,C_{ijk}\left(\overline{\lambda}^{\overline{m}}_A\gamma_\nu\Psi_{B|\mu}\right)^-\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\\
& +\frac{\mathrm{i}}{2}L^\Lambda L^\Sigma\left(\overline{\Psi}^A_\mu\Psi^B_\nu\right)^-\overline{\zeta}_\alpha\gamma^{\mu\nu}\zeta_\beta\,\epsilon_{AB}\,\mathbb{C}^{\alpha\beta}\\
& +\frac{1}{2}L^\Lambda\overline{f}^\Sigma_{i^\star}\left(\overline{\lambda}^{i^\star}_A\gamma^\nu\Psi^\mu_B\right)^-\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta\,\epsilon^{AB}\mathbb{C}^{\alpha\beta}\\
& +\frac{\mathrm{i}}{64}\,C_{ijk}\,C_{lmn}g^{k\overline{r}}\,g^{n\overline{s}}\,\overline{f}^\Lambda_{\overline{r}}\,\overline{f}^\Sigma_{\overline{s}}\,\overline{\lambda}^{iA}\,\gamma_{\mu\nu}\,\lambda^{jB}\,\overline{\lambda}^{kC}\,\gamma^{\mu\nu}\,\lambda^{lD}\,\epsilon_{AB}\epsilon_{CD}\\
& +\frac{\mathrm{i}}{16}\,L^\Lambda\nabla_i f^\Sigma_j\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta\,\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\,\epsilon_{AB}\,\mathbb{C}^{\alpha\beta}\\
& -\frac{\mathrm{i}}{16}\,L^\Lambda\,L^\Sigma\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta\,\overline{\zeta}_\gamma\gamma^{\mu\nu}\zeta_\delta\,\mathbb{C}^{\alpha\beta}\,\mathbb{C}^{\gamma\delta}\Big]
\end{aligned}
$$